

ON SPECIAL LAGRANGIAN FIBRATIONS IN GENERIC TWISTOR FAMILIES OF K3 SURFACES

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ABSTRACT. Filip showed that there are constants $C > 0$ and $\delta > 0$ such that the number of special Lagrangian fibrations of volume $\leq V$ in a generic twistor family of K3 surfaces is $C \cdot V^{20} + O(V^{20-\delta})$.

In this note, we show that δ can be taken to be any number $0 < \delta < \frac{4}{692871}$.

1. INTRODUCTION

The quadratic asymptotics results for the number of maximal cylinders of closed trajectories of lengths $\leq L$ in translation surfaces motivated Filip's study [3] of the counting problem for special Lagrangian (sLag) fibrations with volume $\leq V$ in twistor families of K3 surfaces.

In this direction, Theorem A in Filip's paper [3] ensures the existence of a constant $\delta > 0$ such that the number $N(V)$ of sLag fibrations with volume $\leq V$ in a generic twistor family of K3 surfaces is

$$(1.1) \quad N(V) = C \cdot V^{20} + O(V^{20-\delta})$$

where $C > 0$ is the ratio of volumes of two concrete homogenous spaces.

The goal of this note is to prove that δ can be taken to be $\left(\frac{4}{692871}\right)^-$:

Theorem 1.1. *In the same setting as above, one actually has*

$$N(V) = C \cdot V^{20} + O_\varepsilon(V^{\frac{13857416}{692871} + \varepsilon})$$

for all $\varepsilon > 0$.

2. REDUCTION OF THEOREM 1.1 TO DYNAMICS IN HOMOGENOUS SPACES

Filip derived his counting formula (1.1) from certain equidistribution results. More precisely, let $\Lambda_{\mathbb{Z}}$ be a lattice isomorphic to $H^2(S, \mathbb{Z})$, where S is a K3 surface. Fix $P \subset \Lambda_{\mathbb{R}}$ a positive-definite 3-plane satisfying the genericity condition in [3, Definition 3.1.3]. Denote by $\Lambda_{\mathbb{Z}}^0$ the set of primitive isotropic integral vectors and fix $e \in \Lambda_{\mathbb{Z}}^0$. For each $v \in \Lambda_{\mathbb{R}} = P \oplus P^\perp$, let $v := (v)_P \oplus (v)_{P^\perp}$ with $(v)_P \in P$ and $(v)_{P^\perp} \in P^\perp$. Consider the orthogonal group $G := O(\Lambda_{\mathbb{R}})$, the lattice $\Gamma := O(\Lambda_{\mathbb{Z}})$ and the maximal compact subgroup $K := O(P) \times O(P^\perp)$ of G , and, for a fixed $e \in \Lambda_{\mathbb{Z}}^0$, denote by $H_e := \text{Stab}_G(e)$ and $\Gamma_e = \text{Stab}_\Gamma(e)$.

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The volumes of the locally homogenous spaces $X := \Gamma \backslash G$ and $Y := \Gamma_e \backslash H_e$ are finite. As it is observed in [3, pp. 4], the constants $C > 0$ and $\delta > 0$ in (1.1) are the constant described in [3, Theorem 3.1.6]. In particular,

$$C = \frac{\text{Vol } Y}{20 \cdot \text{Vol } X}$$

The constant $\delta > 0$ is related to the dynamics of a certain one-parameter subgroup a_t of $G \simeq SO(3, 19)(\mathbb{R})$. More concretely, given e and P as above, let e' be the isotropic vector given by

$$e' := (e)_P \oplus -(e)_{P^\perp} \quad \text{where} \quad e := (e)_P \oplus (e)_{P^\perp}$$

In this context, we denote by $\{a_t\}_{t \in \mathbb{R}} \subset G$ the one-parameter subgroup defined as

$$a_t \cdot e = \exp(-t) \cdot e, \quad a_t \cdot e' = \exp(t) \cdot e', \quad a_t|_{(e \oplus e')^\perp} = \text{id}.$$

It is explained in [3, Subsection 3.6] that¹ the quantity δ in (1.1) is

$$(2.1) \quad \delta = \frac{\delta_0}{d_{l_0} + 1}$$

where $(d_l)_{l \in \mathbb{N}}$ are the exponents in [3, Proposition 3.5.10 (ii)], and $\delta_0 > 0$, $l_0 \in \mathbb{N}$ are the constants in the following equidistribution statement in [3, Theorem 4.3.1]:

$$(2.2) \quad \int_{Y_{a_t}} w \, d\mu_{Y_{a_t}} = \frac{\text{Vol } Y}{\text{Vol } X} \int_X w \, d\mu_X + O(\|w\|_l e^{-\delta_0 t})$$

for all Sobolev scales $l \geq l_0$ (see [3, §4.2.2] for the definition of the Sobolev norms in this context).

A quick inspection of the proof of [3, Proposition 3.5.10 (ii)] (related to the thickening of K) reveals that the exponents d_l depend linearly on l . In fact, the constant $c_1(l)$ in [3, Equation (3.5.15)] gives the power of ε associated to the volumes of ε -balls at the origin of $\mathfrak{p}_m \times \mathfrak{n}^+ \times \mathfrak{a}$, that is, $c_1(l) = \dim(G) - \dim(K)$ (and, hence, $c_1(l)$ depends of l). Since the l -th derivative of χ_ε is bounded by a multiple of $\varepsilon^{-c_1(l)-l}$ and it is supported in a ε -neighborhood of K , the l -Sobolev norm of χ_ε is bounded by a multiple of $\varepsilon^{-l-c_1(l)/2}$. Therefore,

$$(2.3) \quad d_l := l + \frac{\dim(G) - \dim(K)}{2}.$$

3. EQUIDISTRIBUTION AND RATES OF MIXING

The constants $\delta_0 > 0$ and $l_0 \in \mathbb{N}$ in (2.2) are described in [3, pp. 40] and they are related to the geometry of $Y \subset X$ and the rate of mixing of a_t .

¹Indeed, [3, pp. 34] says that the optimal choice of δ occurs precisely when the terms $\varepsilon e^{20T} = e^{(20-\delta)T}$ and $\varepsilon^{-d_{l_0}} e^{(20-\delta_0)T} = e^{(20-\delta_0+\delta d_{l_0})T}$ have the same order in T .

3.1. Injectivity radius. We denote by $\text{inj}(x)$ the local injectivity radius at a point $x \in X$ and we let $Y_\varepsilon := \{y \in Y : \text{inj}(y) \geq \varepsilon\}$. By [3, Proposition 4.1.3], we know that the arguments of [1, Lemma 11.2] provide a constant $p > 0$ such that $\mu_Y(Y \setminus Y_\varepsilon) = O(\varepsilon^p)$. Actually, a close inspection of these arguments (of integration over Siegel sets) reveal that $p = 1$ in our specific setting (of $G \simeq SO(3, 19)(\mathbb{R})$):

$$(3.1) \quad \mu_Y(Y \setminus Y_\varepsilon) = O(\varepsilon)$$

3.2. Thickening of Y . Let us fix some parameter $0 < p' < 1$ (very close to one in practice) and consider [3, Proposition 4.1.6] (of thickening of Y) where it is constructed a family of smooth versions ϕ_ε of the characteristic function of Y . As it turns out, ϕ_ε is the product of two functions: τ_ε is a bump function supported² on $Y_{\varepsilon^{p'}}$ and ρ_ε is a bump function supported on the ε -neighborhood of the identity in a certain Lie group N' of dimension $\dim(N') = \dim(X) - \dim(Y)$.

The bump function ρ_ε is obtained by rescaling of a fixed smooth bump function on N' , so that its l -th Sobolev norm satisfies $\|\rho_\varepsilon\|_l = O(\varepsilon^{-l - \frac{\dim(X) - \dim(Y)}{2}})$.

The function τ_ε is

$$\tau_\varepsilon = \frac{\sum_{y_j \in \mathcal{F}} \beta_{y_j, \varepsilon}}{\sum_{y_i \in \mathcal{G}} \beta_{y_i, \varepsilon}} := \frac{\sum_{y_j \in \mathcal{F}} \beta_{y_j, \varepsilon}}{\beta_{\mathcal{G}, \varepsilon}}$$

where $\{y_k\} \subset Y_{\varepsilon^{p'}}$ is a maximal collection of points such that the balls $B(y_k, \varepsilon^3) \subset Y$ are mutually disjoint, $\mathcal{F} = \{y_k\} \cap Y_{4\varepsilon^{p'}}$, $\mathcal{G} = \{y_k\} \cap Y_{2\varepsilon^{p'}}$, and the functions $\beta_{\cdot, \varepsilon}$ are translates of a bump function β_ε whose l -th Sobolev norm is $\|\beta_\varepsilon\|_l = O(\varepsilon^{-l + \frac{\dim(Y)}{2}})$.

On one hand, since a ball B of radius ε at a point of $Y_{\varepsilon^{p'}}$ has volume $O(\varepsilon^{\dim(Y)})$, the cardinality of $\mathcal{G} \cap B$ is $O(\varepsilon^{-2 \dim(Y)})$, the arguments in [1, pp. 1928] imply that the L^∞ -norm of the first l derivatives of $1/\beta_{\mathcal{G}, \varepsilon}$ is $O(\varepsilon^{-l - 2 \dim(Y)})$. On the other hand, the cardinality of \mathcal{F} is $O(\varepsilon^{-3 \dim(Y)})$ and $\|\beta_{y_j, \varepsilon}\|_l = \|\beta_\varepsilon\|_l$. It follows that

$$\|\tau_\varepsilon\|_l = O(\varepsilon^{-l - \frac{9 \dim(Y)}{2}}).$$

By inserting these facts into the definition of ϕ_ε in [3, Equation (4.1.7)], we deduce from Sobolev's lemma that

$$(3.2) \quad \|\phi_\varepsilon\|_l = O(\varepsilon^{-2l - 4 \dim(Y) - \frac{\dim(X)}{2}}),$$

for all $l > \dim(X)/2$, that is, the constant C_l in [3, Proposition 4.1.6 (iii)] is

$$C_l := 2l + 4 \dim(Y) + \frac{\dim(X)}{2}$$

²In fact, Filip sets $p' = 1/2$ for his construction of τ_ε , but any value of $0 < p' < 1$ can be taken here: indeed, the construction of τ_ε can be made as soon as the local product structure statement [3, Proposition 4.1.5] holds (and this is the case for any choice of $0 < p' < 1$ because $\varepsilon^{p'} \gg 2\varepsilon$ for all sufficiently small $\varepsilon > 0$).

For later use, notice that ϕ_ε verifies $\int_X \phi_\varepsilon d\mu_X = \text{Vol } Y + O(\text{Vol}(Y \setminus Y_{\varepsilon^{p'}}))$. By combining this estimate with (3.1), we get

$$(3.3) \quad \int_X \phi_\varepsilon d\mu_X = \text{Vol } Y + O(\varepsilon^{p'})$$

3.3. Wavefront lemma. The proof of Lemma 4.1.10 in [3] says that

$$\int_X w \cdot (\phi_\varepsilon \cdot a_t) d\mu_X = \int_Y w(y a_t) d\mu_Y(y) + O(\varepsilon \text{Lip}(w)) + O(\varepsilon^{pp'} \|w\|_{L^\infty})$$

where $p > 0$ is the parameter such that $\mu_Y(Y \setminus Y_{\varepsilon^{p'}}) = O(\varepsilon^{pp'})$. Therefore, we deduce from (3.1) and Sobolev's lemma that

$$(3.4) \quad \int_X w \cdot (\phi_\varepsilon \cdot a_t) d\mu_X = \int_Y w(y a_t) d\mu_Y(y) + O(\varepsilon^{p'} \|w\|_l)$$

for all $l > 1 + \dim(X)/2$.

3.4. Reduction of equidistribution to rate of mixing. By following [3, pp. 40], let us compute the constants $\delta_0 > 0$ and $l_0 \in \mathbb{N}$ in (2.2) in terms of the following quantitative mixing statement: there exists $\delta'_0 > 0$ such that

$$(3.5) \quad \left| \int_X \alpha \cdot (\beta \cdot g) d\mu - \left(\int_X \alpha d\mu \right) \left(\int_X \beta d\mu \right) \right| = O(\|\alpha\|_l \|\beta\|_l \|g\|^{-\delta'_0})$$

for all $l \geq l'_0$. (Here, $\mu = \mu_X / \text{Vol } X$ is the normalized Haar measure.)

For this sake, we observe that (3.5) says that

$$\int_X w \cdot (\phi_\varepsilon \cdot a_t) d\mu_X = \frac{1}{\text{Vol } X} \left(\int_X w d\mu_X \right) \left(\int_X \phi_\varepsilon d\mu_X \right) + O(\|w\|_l \|\phi_\varepsilon\|_l e^{-t\delta'_0})$$

for $l \geq l'_0$.

By (3.2) and (3.3), the previous estimate implies

$$\int_X w \cdot (\phi_\varepsilon \cdot a_t) d\mu_X = \frac{\text{Vol } Y}{\text{Vol } X} \left(\int_X w d\mu_X \right) + O(\varepsilon^{p'} \|w\|_l) + O(\|w\|_l \varepsilon^{-C_l} e^{-t\delta'_0})$$

for all $l \geq \max\{l'_0, \lfloor \dim(X)/2 \rfloor + 1\}$.

By plugging (3.4) into the estimate above, we conclude that

$$\int_{Y_{a_t}} w d\mu_{Y_{a_t}} = \frac{\text{Vol } Y}{\text{Vol } X} \left(\int_X w d\mu_X \right) + O(\varepsilon^{p'} \|w\|_l) + O(\|w\|_l \varepsilon^{-C_l} e^{-t\delta'_0})$$

for all $l \geq l_0 := \max\{l'_0, \lfloor \dim(X)/2 \rfloor + 2\}$.

By taking $\varepsilon := e^{-\delta''_0 t}$ and by optimizing³ the value of δ''_0 , we obtain that

$$\int_{Y_{a_t}} w d\mu_{Y_{a_t}} = \frac{\text{Vol } Y}{\text{Vol } X} \left(\int_X w d\mu_X \right) + O(\|w\|_{l_0} e^{-t\delta_0})$$

for $l_0 := \max\{l'_0, \lfloor \dim(X)/2 \rfloor + 2\}$ and $\delta_0 := \frac{p'}{p' + C_{l_0}} \delta'_0$.

³I.e., we choose $\delta''_0 > 0$ so that $\varepsilon^{p'} = \varepsilon^{-C_{l_0}} e^{-t\delta'_0}$.

Since $0 < p' < 1$ is an arbitrary parameter, we deduce that (2.2) holds for $l_0 := \max\{l'_0, \lfloor \dim(X)/2 \rfloor + 2\}$ and any choice of

$$(3.6) \quad 0 < \delta_0 < \frac{\delta'_0}{1 + 2l_0 + 4 \dim(Y) + \frac{\dim(X)}{2}}$$

4. RATES OF MIXING AND REPRESENTATION THEORY

Definition 4.1. 1. A unitary representation π of G in a (separable) Hilbert space \mathcal{H}_π is a morphism $G \rightarrow \mathrm{U}(\mathcal{H}_\pi)$ such that for any $v \in \mathcal{H}_\pi$ the map $G \rightarrow \mathcal{H}_\pi$; $g \mapsto \pi(g)v$ is continuous. If this map is smooth one says that v is a C^∞ -vector of π . We denote by \mathcal{H}_π^∞ the set of C^∞ -vectors of π .

2. Given two vectors $v, w \in \mathcal{H}_\pi$, we define the *matrix coefficient* $c_{v,w} : G \rightarrow \mathbb{C}$ of π as the continuous map $g \mapsto \langle \pi(g)v, w \rangle$. The coefficient $c_{v,w}$ is said to be *K-finite* if both the vector spaces generated by $\pi(K) \cdot v$ and $\pi(K) \cdot w$ are finite dimensional.

3. Let $p(\pi)$ be the infimum of the set of real numbers $p \geq 2$ such that all K -finite matrix coefficients of π are in $L^p(G)$.

4. Say that a unitary representation σ of G is *weakly contained* in π if any matrix coefficient of σ can be obtained as the limit, with respect to the topology of uniform convergence on compact subsets, of a sequence of matrix coefficients of π .

Given an element $g = nak \in G$, we write $a = e^{H(g)}$. The *Harish-Chandra* function is $\Xi = \Xi_G : G \rightarrow \mathbb{R}$ defined by

$$\Xi(g) = \int_K e^{-\rho(H(kg^{-1}))} dk$$

where ρ is half the sum of the positive restricted roots counting multiplicities. The Harish-Chandra function decreases exponentially fast along A^+ ; modulo a logarithmic factor, it decreases like $e^{-\rho(H)}$.

Let $d = \dim(K)$ be the dimension of K and fix a basis \mathcal{B} of the Lie algebra \mathfrak{k} of K . Given a smooth vector $v \in \mathcal{H}_\pi^\infty$ we set

$$S(v) = \sum_{\mathrm{ord}(D) \leq \lfloor d/2 \rfloor + 1} \|\pi(D)v\|,$$

where D varies among all monomials in elements of \mathcal{B} of degree $\leq \lfloor d/2 \rfloor + 1$ and, if X_1, \dots, X_r are elements of \mathcal{B} , we have $\pi(X_1 \cdots X_r) = \pi(X_1) \cdots \pi(X_r)$ and each $\pi(X_i)$ acts by derivation.

Proposition 4.2. *For all positive ε and $k \in \mathbb{N}^*$, there exists a constant $C = C(\varepsilon, k)$ such that if π is a unitary representation of G with $p(\pi) \leq 2k$, then for all $v, w \in \mathcal{H}_\pi^\infty$ and for all positive t we have:*

$$(4.1) \quad |\langle \pi(a_t)v, w \rangle| \leq CS(v)S(w)e^{-(p/k-\varepsilon)t},$$

where $p = \rho(H)$ and H is the infinitesimal generator of the one-parameter subgroup (a_t) .

Proof. Up to replacing π by the tensor product $\pi^{\otimes k}$ we may suppose that $k = 1$; see [2, p. 108]. It then follows from [2, Theorem 1] that π is weakly contained in the (right) regular representation $L^2(G)$. We are then reduced to prove the proposition in the case where π is the regular representation of G (and $k = 1$); see the proof of [2, Theorem 2] for more details on this last reduction.

Now consider v and w in $L^2(G) \cap C^\infty(G)$. The functions

$$\varphi : x \mapsto \sup_{k \in K} |v(xk)| \text{ and } \psi : x \mapsto \sup_{k \in K} |w(xk)|$$

are both positive and K -invariant, and we have:

$$|\langle \pi(a_t)v, w \rangle_{L^2(G)}| \leq \int_G \varphi(xa_t)\psi(x)dx = |\langle \pi(a_t)\varphi, \psi \rangle_{L^2(G)}|.$$

Now the Sobolev lemma (see [5, Proposition 2.6]) implies that the L^∞ norms of φ and ψ can be estimated in terms of their Sobolev norms along K . More precisely: there exists a constant c such that for all $x \in G$,

$$\varphi(x)^2 = \sup_{k \in K} |v(xk)|^2 \leq c \sum_{\text{ord}(D) \leq [d/2] + 1} \|(\rho(D)v)(x \cdot)\|_{L^2(K)}.$$

Integrating over G (here we assume for simplicity that the measure of K is 1) one concludes that $\|\varphi\|_{L^2(G)} \leq \sqrt{c}S(v)$ and similarly for ψ . It remains to prove that there exists a constant d_ε such that if $\varphi, \psi \in L^2(G)$ are two K -invariant, positive functions of norm 1, then

$$|\langle \pi(a_t)\varphi, \psi \rangle| \leq d_\varepsilon e^{-(p/k+\varepsilon)t}.$$

First it follows from the computations of [2, pp. 106-107] that

$$\begin{aligned} |\langle \pi(g)\varphi, \psi \rangle| &= \int_K \left(\int_{NA} \varphi(na)\psi(nakg^{-1})e^{2\rho(\log a)}dn da \right) dk \\ &\leq \|\varphi\| \int_K \left(\int_{NA} \psi(naH(kg^{-1}))^2 e^{2\rho(\log a)}dn da \right)^{1/2} dk \\ &= \|\varphi\| \cdot \|\psi\| \int_K e^{-\rho(H(kg^{-1}))} dk = \Xi(g). \end{aligned}$$

Now recall that, up to “logarithmic factors”, the function $\Xi(a_t)$ decreases like $e^{-t\rho(H)} = e^{-pt}$. The proposition follows. \square

We shall apply this proposition to the (quasi-)regular representation π of G in the subspace $L_0^2(\Gamma \backslash G)$ of $L^2(\Gamma \backslash G)$ that is orthogonal to the space of constant functions. It follows from [4] that $p(\pi) = 20$. Proposition 4.2 therefore applies with $k = 10$. Note that in our case $p = 10$.

Now let α and β be two smooth functions in $L^2(X)$ then

$$\alpha_0 := \alpha - \int_X \alpha d\mu \text{ and } \beta_0 := \beta - \int_X \beta d\mu \in L_0^2(X)$$

and we have:

$$\langle \pi(g)\alpha_0, \beta_0 \rangle_{L_0^2(X)} = \int_X \alpha \cdot (\beta \cdot g) d\mu - \left(\int_X \alpha d\mu \right) \left(\int_X \beta d\mu \right).$$

From Proposition 4.2 and the fact that $S(\alpha) \leq \|\alpha\|_{[d/2]+1}$ we conclude that

$$\left| \int_X \alpha \cdot (\beta \cdot a_t) d\mu - \left(\int_X \alpha d\mu \right) \left(\int_X \beta d\mu \right) \right| = O(\|\alpha\|_l \|\beta\|_l e^{-t\delta'_0})$$

for any $l \geq l'_0 := \lfloor \dim(K)/2 \rfloor + 1$ and any $\delta'_0 < 1$.

5. END OF PROOF OF THEOREM 1.1

The explicit value of δ announced in Theorem 1.1 can be easily derived from the discussion above. Indeed, we just saw in Section 4 that $\delta'_0 = 1-$ and $l'_0 = \lfloor \dim(K)/2 \rfloor + 1$. Because $174 = \dim(K) < \dim(X) = 231$ and $\dim(Y) = 210$, we deduce from (3.6) that $l_0 = \lfloor \dim(X)/2 \rfloor + 2 = 117$ and

$$\delta_0 = \left(\frac{1}{1 + 2 \times 117 + 4 \times 210 + \frac{231}{2}} \right)^- = \left(\frac{2}{2381} \right)^-$$

Finally, by inserting these informations into (2.3) and (2.1), we conclude that

$$\delta = \frac{\delta_0}{l_0 + \frac{57}{2}} = \left(\frac{4}{692871} \right)^- \approx (5.7730804146 \dots \times 10^{-6})^-.$$

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